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Scattering and bound states for a class of non-central potentials

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Abstract

We obtain L^2-series solutions of the three-dimensional Schrödinger wave equation for a large class of non-central potentials that includes, as special cases, the Aharonov–Bohm, Hartmann and magnetic monopole potentials. It also includes contributions of the potential term, $\cos\theta/r^2$ (in spherical coordinates). The solutions obtained are for all energies, the discrete (for bound states) as well as the continuous (for scattering states). The L^2 bases of the solution space are chosen such that the matrix representation of the wave operator is tridiagonal. The expansion coefficients of the radial and angular components of the wavefunction are written in terms of orthogonal polynomials satisfying three-term recursion relations resulting from the matrix wave equation.

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1. Introduction

By relaxing the constraint of a diagonal representation of the Hamiltonian and allowing for the next higher level of generalization, which is that of tridiagonal matrices, we found a larger solution space for the wave equation with an extended class of exactly solvable potentials [1]. The L^2 series solutions obtained as such include the discrete (for bound states) as well as the continuous (for scattering states) spectrum of the Hamiltonian. Due to the tridiagonal structure of the matrix wave equation the problem, in this approach, translates into finding solutions of the resulting three-term recursion relation for the expansion coefficients of the wavefunction. These are written in terms of orthogonal polynomials, some of which are well known but some are new while others are modified versions of known polynomials. In a recent article [1], we obtained solutions of problems in one and three dimensions using this approach. The solutions of some of the classic problems such as the Coulomb and Morse were reproduced adding, however, new tridiagonal representations to the solution space.

We also found generalizations of others, such as the Hulthén problem, where we obtained an extended class of solutions and introduced their associated orthogonal polynomials. This kind of development embodies powerful tools in the analysis of solutions of the wave equation by exploiting the intimate connection and interplay between tridiagonal matrices and the theory of orthogonal polynomials. In such analysis, one is at liberty to employ a wide range of well established methods and numerical techniques associated with these settings such as quadrature approximation and continued fractions. These formulations were also extended to the study of the relativistic problem. The Dirac–Coulomb and Dirac–Morse are two relativistic problems, beside others, that have already been worked out using this approach [2].

In this paper, we investigate the nonrelativistic problem in three dimensions with non-central potentials using the same approach and obtain scattering and bound states solutions of the Schrödinger wave equation. We consider the time-independent potential of the form $V(\vec{r}) = V(r, \theta)$ such that it is separable in spherical coordinates. Specifically, we study the following class of potentials (in the atomic units, $\hbar = \mathcal{M} = e = 1$, where $\mathcal{M}$ is the particle's mass and e is its electric charge)

$$V(r, \theta) = V(r) + \frac{1}{2r^2}\left(\frac{\hat{C} + C\cos\theta}{\sin^2\theta} - C_0\cos\theta\right), \tag{1.1}$$

where $\hat{C}$, C and C_0 are real potential parameters. The Aharonov–Bohm [3] and Hartmann [4] potentials are special cases for which $C = C_0 = 0$ (for the pure Aharonov–Bohm effect, $\hat{C}$ is discrete via its linear dependence on an integer $m \in \mathbb{Z}$ which comes from the phase quanta of the angular component of the wavefunction, $e^{im\phi}$). The case where $C = \pm\hat{C}$ and $C_0 = 0$ corresponds to the magnetic monopole potential with singularity along the $\pm z$ axis [5]. Our main contribution to the solution of this kind of problem is twofold. The first is the introduction of the three-dimensional potential term $\frac{\cos\theta}{r^2}$ which, to the best of our knowledge, was never treated exactly before. The second is the *simultaneous analytic* solution of scattering and bound states in the same formulation. We take the Coulomb interaction as the radial component of the potential. That is, we take $V(r) = \mathcal{Z}/r$, where $\mathcal{Z}$ is the electric charge coupling. Additionally, we consider briefly in section 7 the radial oscillator potential $V(r) = \frac{1}{2}\omega^4 r^2$, where ω is the oscillator frequency.

For an introduction to the above-mentioned approach and its implementation on some examples in one and three dimensions (with spherical symmetry) one may consult the papers in [1, 2]. Nonetheless, it might be useful to give, in few lines, a brief account as follows. Let $\{\varphi_n(\vec{r})\}_{n=0}^{\infty}$ be a complete set of L^2 basis in the configuration space with coordinates $\vec{r}$ that supports a tridiagonal matrix representation for the wave operator. That is, by expanding the wavefunction as $|\psi(\vec{r}, E)\rangle = \sum_n f_n(E)|\varphi_n(\vec{r})\rangle$, the matrix representation of the wave operator in this basis could be written as follows:

$$\langle\varphi_n|H - E|\varphi_m\rangle = (a_n - z)\delta_{n,m} + b_n\delta_{n,m-1} + b_{n-1}\delta_{n,m+1}, \tag{1.2}$$

where z and the coefficients $\{a_n, b_n\}_{n=0}^{\infty}$ are real and, in general, functions of the energy, E, angular momentum and potential parameters. Therefore, the matrix wave equation, which is obtained by expanding $|\psi\rangle$ in $(H - E)|\psi\rangle = 0$ as $\sum_m f_m|\varphi_m\rangle$ and projecting on the left by $\langle\varphi_n|$, results in the following three-term recursion relation:

$$z f_n = a_n f_n + b_{n-1} f_{n-1} + b_n f_{n+1}. \tag{1.3}$$

Consequently, the problem translates into finding solutions of this recursion relation for the expansion coefficients of the wavefunction. In most cases, this recurrence relation could be solved easily and directly by correspondence with those for well-known orthogonal

polynomials. Moreover, equation (1.2) shows that the discrete energy spectrum is easily obtained by imposing the diagonalization constraint which requires that

$$b_n = 0, \qquad a_n - z = 0. \tag{1.4}$$

for all n. One could also obtain another ('irregular') solution to the recursion (1.3) by starting with a different initial (seed) relation (for $n = 0$). The solution associated with the homogeneous recursion relation that starts with

$$(a_0 - z) f_0 + b_0 f_1 = 0 \tag{1.5}$$

is the regular solution. The other is associated with a non-homogeneous recursion whose initial relation (with an overall normalizing constant) is

$$(a_0 - z) f_0 + b_0 f_1 = 1. \tag{1.6}$$

One of these two solutions behaves asymptotically ($n \to \infty$) as sine-like while the other behaves as cosine-like. These two solutions have the same asymptotic limits as the regular and irregular solutions of the second-order differential wave equation. Phase shift and cross section could be obtained algebraically by studying these asymptotic limits. These are issues of concern in algebraic scattering theories such as the J-matrix method [6]. In the present work, however, we will only be concerned with the regular solutions of the wave equation.

We start in the following section by formulating the problem and writing down the basis elements for the angular and radial components of the wavefunction that support a tridiagonal matrix representation for the associated wave operator. The corresponding components of the non-central separable potential that are compatible with the tri-diagonal representations are also obtained. Explicit construction of the angular wavefunction for three possible configurations is given in section 3. The radial component, which is compatible with the Coulomb interaction, is obtained in section 4. The complete solution space splits into two disconnected subspaces: one for $C_0 \neq 0$ and another for $C_0 = 0$. These are constructed in sections 5 and 6, respectively. Additionally, in section 5, we give the complete and explicit solution for the special case where $\hat{C} = C = 0$ and $C_0 \neq 0$. This case is particularly unique and interesting since the non-spherical three-dimensional potential, $\frac{\cos\theta}{r^2}$, has never been treated analytically before (neither for its bound states nor for its scattering states). In section 7, we study briefly the case where the radial component of the non-central potential is that of the spherical oscillator. The new orthogonal polynomials associated with the tridiagonal representation of the angular component of the solution space will be investigated in appendix B. The resolvent operator and weight (density) function for these polynomials are obtained in terms of the recursion coefficients $\{a_n, b_n\}$. Finally, in appendix C we formulate the problem of a charged particle moving in a cylindrical electromagnetic vector potential (e.g., outside an infinitely long and thin current solenoid) and establish its connection to the present problem. As an example, we obtain the bound states solution for the combined Aharonov–Bohm effect and a magnetic monopole.

2. Non-central separable potentials in spherical coordinates

In the atomic units $\hbar = \mathcal{M} = e = 1$, the time-independent Schrödinger wave equation for a structureless scalar particle in a potential $V(\vec{r})$ is

$$\left[-\tfrac{1}{2}\vec{\nabla}^2 + V(\vec{r}) - E\right]\psi = 0, \tag{2.1}$$

where $\vec{\nabla}$ is the three-dimensional Laplacian. The energy, E, is real and it is either discrete for bound states, or continuous for scattering states. In the spherical coordinates, $\vec{r} = \{r, \theta, \phi\}$, this wave equation could be written explicitly as follows:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left[(1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + \frac{1}{1-x^2}\frac{\partial^2}{\partial \phi^2}\right] - 2V + 2E\right\}\psi = 0, \tag{2.2}$$

where $x = \cos\theta$. Consequently, this equation is separable for potentials of the form

$$V(\vec{r}) = V_r(r) + \frac{1}{r^2}\left[V_\theta(x) + \frac{1}{1-x^2}V_\phi(\phi)\right]. \tag{2.3}$$

This is so because if we write the wavefunction as $\psi(r, \theta, \phi) = r^{-1}R(r)\Theta(\theta)\Phi(\phi)$, then the wave equation (2.2) with the potential (2.3) gets separated in all three coordinates as follows:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} - 2V_\phi + 2E_\phi\right)\Phi = 0, \tag{2.4a}$$

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2E_\phi}{1-x^2} - 2V_\theta + 2E_\theta\right]\Theta = 0, \tag{2.4b}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2E_\theta}{r^2} - 2V_r + 2E\right)R = 0, \tag{2.4c}$$

where E_ϕ and E_θ are the separation constants, which are real and dimensionless. Square integrability of the basis is with respect to the following integration measures:

$$\int |\psi|^2\,\mathrm{d}^3\vec{r} = \int_0^\infty |R|^2\,\mathrm{d}r \int_{-1}^{+1} |\Theta|^2\,\mathrm{d}x \int_0^{2\pi} |\Phi|^2\,\mathrm{d}\phi. \tag{2.5}$$

They are also required to satisfy the boundary conditions that $R(0) = R(\infty) = 0$, $\Phi(\phi) = \Phi(\phi + 2\pi)$, $\Theta(0)$ and $\Theta(\pi)$ are finite. If we specialize to the case where $V_\phi = 0$, then the normalized solution of equation (2.4a) that satisfies the boundary conditions is

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}m\phi}, \qquad m = 0, \pm1, \pm2, \ldots, \tag{2.6}$$

giving $E_\phi = \frac{1}{2}m^2$.

The L^2 basis elements for the angular wavefunction component $\Theta(\theta)$ in the configuration space with coordinate $x \in [-1, +1]$ that satisfy the boundary conditions are

$$\chi_n(x;\mu,\nu) = A_n(1-x)^\alpha(1+x)^\beta P_n^{(\mu,\nu)}(x), \tag{2.7}$$

where $x = \cos\theta$, $P_n^{(\mu,\nu)}(x)$ is the Jacobi polynomial of order n and $n = 0, 1, 2, \ldots$ The dimensionless real parameters $\alpha, \beta > 0$, $\mu, \nu > -1$ and A_n is the normalization constant

$$A_n = \sqrt{\frac{2n+\mu+\nu+1}{2^{\mu+\nu+1}}\frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}}. \tag{2.8}$$

Using the differential equation (A.3) and differential formula (A.4) for the Jacobi polynomials, shown in appendix A, we obtain

$$\begin{aligned}\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x}\right]\chi_n = {} & \left[-n\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right)\left(\frac{\mu-2\alpha}{1-x} + \frac{2\beta-\nu}{1+x}\right)\right. \\ & \left. + \alpha^2\frac{1+x}{1-x} + \beta^2\frac{1-x}{1+x} - (2\alpha\beta+\alpha+\beta) - n(n+\mu+\nu+1)\right]\chi_n \\ & + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}\left(\frac{\mu-2\alpha}{1-x} + \frac{2\beta-\nu}{1+x}\right)\frac{A_n}{A_{n-1}}\chi_{n-1}.\end{aligned} \tag{2.9}$$

Therefore, the action of the differential operator of equation (2.4*b*) on the basis element (2.7) reads as follows:

$$(H_\theta - E_\theta)\chi_n = \left[\frac{n}{2}\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right)\left(\frac{\mu-2\alpha}{1-x} + \frac{2\beta-\nu}{1+x}\right) - \frac{\alpha^2}{2}\frac{1+x}{1-x} - \frac{\beta^2}{2}\frac{1-x}{1+x} + \frac{E_\phi}{1-x^2} + V_\theta - E_\theta + \alpha\beta + \frac{\alpha+\beta}{2} + \frac{n}{2}(n+\mu+\nu+1)\right]\chi_n - \frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}\left(\frac{\mu-2\alpha}{1-x} + \frac{2\beta-\nu}{1+x}\right)\frac{A_n}{A_{n-1}}\chi_{n-1}. \tag{2.10}$$

The recurrence relation (A.1) and orthogonality relation (A.5) of the Jacobi polynomials show that a tridiagonal matrix representation, $\langle\chi_n|H_\theta - E_\theta|\chi_{n'}\rangle$, is obtainable only for a limited number of special choices of potential components V_θ and for specific relations among the parameters as follows:

(1) $\alpha = \dfrac{\mu}{2}$, $\beta = \dfrac{\nu}{2}$ and $V_\theta = \dfrac{1}{2}\dfrac{\frac{1}{2}(\mu^2+\nu^2) - m^2 + \frac{1}{2}(\mu^2-\nu^2)x}{1-x^2} - \dfrac{C_0}{2}x$ (2.11*a*)

(2) $\alpha = \dfrac{\mu}{2}$, $\beta = \dfrac{\nu+1}{2}$ and $V_\theta = \dfrac{1}{2}\dfrac{\frac{1}{2}(\mu^2+C_1) - m^2 + \frac{1}{2}(\mu^2-C_1)x}{1-x^2}$ (2.11*b*)

(3) $\alpha = \dfrac{\mu+1}{2}$, $\beta = \dfrac{\nu}{2}$ and $V_\theta = \dfrac{1}{2}\dfrac{\frac{1}{2}(C_2+\nu^2) - m^2 + \frac{1}{2}(C_2-\nu^2)x}{1-x^2}$ (2.11*c*)

where $m = 0, \pm1, \pm2, \ldots$, and $\{C_i\}_{i=0}^2$ are dimensionless real parameters. The first possibility (2.11*a*) eliminates the χ_{n-1} term from equation (2.10), whereas the last two allow this term to contribute to the matrix elements above and below the diagonal. The details of these three possibilities will be given in the following section.

Now, the radial component of the wavefunction, $R(r)$, could be taken as an element in the space spanned by the L^2 functions

$$\xi_n(y;\lambda,\nu) = B_n y^\alpha \,\mathrm{e}^{-y/2} L_n^\nu(y), \tag{2.12}$$

where $y = \lambda r$ and $L_n^\nu(y)$ is the Laguerre polynomial of order n. The real parameter λ is positive and carries the dimension of inverse length (i.e., it is a length scale parameter). On the other hand, the dimensionless parameters $\alpha > 0$ and $\nu > -1$. It should be understood that the basis parameters α and ν are reused here for the purpose of economy in the use of symbols but are not the same as those that appear in the angular wavefunction basis (2.7). The normalization constant $B_n = \sqrt{\lambda\Gamma(n+1)/\Gamma(n+\nu+1)}$. Using the differential equation (A.8) and differential formula (A.9) for the Laguerre polynomials, we obtain

$$\frac{\mathrm{d}^2\xi_n}{\mathrm{d}r^2} = \lambda^2\left[-\frac{n}{y}\left(1 + \frac{\nu+1-2\alpha}{y}\right) + \frac{\alpha(\alpha-1)}{y^2} - \frac{\alpha}{y} + \frac{1}{4}\right]\xi_n - \lambda^2\frac{(n+\nu)(2\alpha-\nu-1)}{y^2}\frac{A_n}{A_{n-1}}\xi_{n-1}. \tag{2.13}$$

Therefore, the action of the differential wave operator of equation (2.4*c*) on the basis element (2.12) gives the following:

$$(H-E)\xi_n = \frac{\lambda^2}{2}\left[\frac{n}{y}\left(1 + \frac{\nu+1-2\alpha}{y}\right) + \frac{2E_\theta - \alpha(\alpha-1)}{y^2} + \frac{\alpha}{y} - \frac{1}{4} + \frac{2}{\lambda^2}(V_r - E)\right]\xi_n + \frac{\lambda^2}{2}\frac{(n+\nu)(2\alpha-\nu-1)}{y^2}\frac{A_n}{A_{n-1}}\xi_{n-1}. \tag{2.14}$$

The recurrence relation (A.6) and orthogonality relation (A.10) for the Laguerre polynomials show that a tridiagonal matrix representation $\langle \xi_n | H - E | \xi_{n'} \rangle$ is possible only for a limited number of special radial potential components V_r and results in the following two possibilities:

$$(1) \quad \nu = 2\alpha - 1, \qquad \alpha(\alpha - 1) = 2E_\theta \qquad \text{and} \qquad V_r = \frac{\mathcal{Z}}{r} \tag{2.15a}$$

$$(2) \quad \nu = 2\alpha - 2, \qquad \lambda^2 = -8E \qquad \text{and} \qquad V_r = \frac{\mathcal{Z}}{r} + \frac{\mathcal{B}/2}{r^2} \tag{2.15b}$$

where $\mathcal{Z}$ and $\mathcal{B}$ are real potential parameters: $\mathcal{Z}$ is the particle's charge and $\mathcal{B}$ is a centripetal potential barrier parameter. In what follows, we only consider the first case since the second is restricted (for real representations) to negative energies only. On the other hand, by taking the configuration coordinate in (2.12) as $y = (\lambda r)^2$, the basis function becomes compatible with the problem whose radial potential component is that of the oscillator, $V_r = \frac{1}{2}\omega^4 r^2$. This will be discussed briefly in section 7.

3. Solution space for the angular component

3.1. Case (2.11a)

We start by studying the case (2.11a) for which the angular component of the non-central potential could be written as

$$V_\theta = \frac{\hat{C} + C\cos\theta}{2\sin^2\theta} - \frac{C_0}{2}\cos\theta. \tag{3.1}$$

Thus the basis parameters μ and ν are related to the potential parameters $\hat{C}$ and C and to the wavefunction quantum phase number m by

$$\mu_m = \sqrt{m^2 + \hat{C} + C}, \qquad \nu_m = \sqrt{m^2 + \hat{C} - C}. \tag{3.2}$$

These relations show that μ and ν are discrete (indexed by m) and positive. Moreover, they also require that, for real representations, $m^2 \geqslant \max(-\hat{C} \pm C)$. That is, if we define M as the smallest integer greater than $\max(|\hat{C} \pm C|)^{1/2}$, then

$$|m| = \begin{cases} 0, 1, 2, \ldots, & \max(\hat{C} \pm C) \geqslant 0 \\ M, M+1, M+2, \ldots, & \max(\hat{C} \pm C) < 0. \end{cases} \tag{3.3}$$

It is worth noting that a unique solution might exist for $m = \pm(M-1)$ in which the parameter(s) μ_m or (and) ν_m is (are) the negative of that (those) given by equation (3.2), and in the range $(-1, 0)$. This happens for special values of the potential parameters $\hat{C}$ and C satisfying any (both) of the following two inequalities:

$$1 - \hat{C} \pm C > (M-1)^2 > -\hat{C} \pm C. \tag{3.4}$$

Now, to obtain the tridiagonal matrix representation $\langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle$ we employ the orthogonality property (A.5) and recurrence relation (A.1) for the Jacobi polynomials into the action of the differential operator as given by equation (2.10). The result is as follows:

$$\begin{aligned} \langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle = & \left[\frac{(\nu^2 - \mu^2)C_0/2}{(2n+\mu+\nu)(2n+\mu+\nu+2)} + \frac{1}{2}\left(n + \frac{\mu+\nu+1}{2}\right)^2 \right. \\ & \left. - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)^2 \right] \delta_{n,n'} + \frac{C_0}{2n+\mu+\nu}\sqrt{\frac{n(n+\mu)(n+\nu)(n+\mu+\nu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}\,\delta_{n,n'+1} \\ & + \frac{C_0}{2n+\mu+\nu+2}\sqrt{\frac{(n+1)(n+\mu+1)(n+\nu+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}\,\delta_{n,n'-1} \end{aligned} \tag{3.5}$$

where we have introduced the dimensionless real parameter γ by writing $2E_\theta \equiv \gamma(\gamma+1)$. The subscript m on μ and ν was removed for the sake of clarity and simplicity in presentation. For arbitrary values of C_0, equation (3.5) shows that as the integers n and m increase the representation degenerates by changing its signature (becoming indefinite) when crossing the threshold defined by

$$\left(n+\frac{\mu_m+\nu_m+1}{2}\right)^2 \leqslant \left(\gamma+\frac{1}{2}\right)^2. \tag{3.6}$$

To keep the representation, which is bounded from below, definite and prevent it from degenerating we require that the set of integers $\{n\in\mathbb{N}, m\in\mathbb{Z}\}$ satisfy the constraint (3.6). Therefore, for a given γ, $m=0,\pm1,\pm2,\ldots,\pm j$ [or $m=\pm M,\pm(M+1),\pm(M+2),\ldots,\pm j$, see equation (3.3)] such that j is the maximum integer satisfying $\mu_j+\nu_j\leqslant|2\gamma+1|-1$. Moreover, $n=0,1,2,\ldots,N$, where N is the maximum integer satisfying $N\leqslant\left|\gamma+\frac{1}{2}\right|-\frac{\mu_0+\nu_0+1}{2}$ $\left[\text{or, } N\leqslant\left|\gamma+\frac{1}{2}\right|-\frac{\mu_M+\nu_M+1}{2}\right]$. Expanding the angular component of the wavefunction as $\Theta(\theta)=\sum_n f_n(E_\theta)\chi_n(x;\mu,\nu)$, then equations (2.4*b*) and (3.5) give the following three-term recursion relation for the expansion coefficients:

$$\begin{aligned} zf_n = &\left[\frac{\nu^2-\mu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)}+\frac{1}{C_0}\left(n+\frac{\mu+\nu+1}{2}\right)^2\right]f_n \\ &+\frac{2}{2n+\mu+\nu}\sqrt{\frac{n(n+\mu)(n+\nu)(n+\mu+\nu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}\,f_{n-1} \\ &+\frac{2}{2n+\mu+\nu+2}\sqrt{\frac{(n+1)(n+\mu+1)(n+\nu+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}\,f_{n+1} \end{aligned} \tag{3.7}$$

where $z=\frac{1}{C_0}\left(\gamma+\frac{1}{2}\right)^2$ and $C_0\neq0$. This recursion relation could be rewritten in terms of a three-parameter polynomial defined as

$$H_n^\sigma(z;\mu,\nu)=\frac{1}{\sqrt{2n+\mu+\nu+1}}\sqrt{\frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}}\,f_n(z), \tag{3.8}$$

where $\sigma=1/C_0$. In terms of these orthogonal polynomials the recursion relation (3.7) becomes

$$\begin{aligned} zH_n^\sigma = &\left[\frac{\nu^2-\mu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)}+\sigma\left(n+\frac{\mu+\nu+1}{2}\right)^2\right]H_n^\sigma \\ &+\frac{2(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)}H_{n-1}^\sigma+\frac{2(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)}H_{n+1}^\sigma. \end{aligned} \tag{3.9}$$

These polynomials, as far as we know, have not been studied before. However, comparing this recurrence relation with (A.1) for the Jacobi polynomials, we can write

$$P_n^{(\mu,\nu)}(z)=\lim_{\sigma\to0}H_n^\sigma(z;\mu,\nu). \tag{3.10}$$

In this limit, E_θ should be allowed to increase such that the ratio $\left(2E_\theta+\frac{1}{4}\right)/C_0$, which is equal to z, becomes finite and belongs to the interval $[-1,+1]$. The polynomials $H_n^\sigma(z;\mu,\nu)$ are uniquely defined by the recursion (3.9) and the initial normalizing relation

that $H_0^\sigma(z;\mu,\nu)=1$. Now, the angular component of the wavefunction could then be written as the L^2-series sum

$$\Theta_m^a(\theta)=\mathcal{N}_m^a(z)\sum_{n=0}^{N_m}\frac{2n+\mu+\nu+1}{\sqrt{2^{\mu+\nu+1}}}\frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}$$
$$\times H_n^\sigma(z;\mu,\nu)(1-x)^{\frac{\mu}{2}}(1+x)^{\frac{\nu}{2}}P_n^{(\mu,\nu)}(x), \tag{3.11a}$$

where $\mathcal{N}_m^a(z)$ is a normalization constant that depends on z, m and the physical parameters of the problem but, otherwise, independent of n. To make $\Theta_m^a(\theta)$ z-normalizable, we write $\mathcal{N}_m^a(z)=\sqrt{\rho^\sigma(z)}$, where $\rho^\sigma(z)$ is the weight (density) function associated with the polynomials $\{H_n^\sigma\}$:

$$\int\rho^\sigma(z)H_n^\sigma(z;\mu,\nu)H_m^\sigma(z;\mu,\nu)\,\mathrm{d}z=\frac{1}{2n+\mu+\nu}\frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}\delta_{nm}. \tag{3.12}$$

In appendix B we show how to obtain this density function from the resolvent operator (Green's function) associated with these polynomials using the coefficients of the recursion relation, $\{a_n,b_n\}_{n=0}^\infty$. The sum in the series (3.11a) runs from $n=0$ to $n=N_m$ where N_m, for a given m, is obtained from condition (3.6) as the largest integer n satisfying the inequality

$$n\leqslant\left|\gamma+\frac{1}{2}\right|-\frac{\mu_m+\nu_m+1}{2}. \tag{3.6$'$}$$

The subscript m appearing in $\Theta_m^a(\theta)$ is due to the fact that the parameters μ and ν depend on m as shown in equation (3.2).

To obtain the diagonal representation,

$$\Theta_{nm}^a(\theta)=\chi_n(x;\mu_m,\nu_m)=\sqrt{\frac{2n+\mu_m+\nu_m+1}{2^{\mu_m+\nu_m+1}}\frac{\Gamma(n+1)\Gamma(n+\mu_m+\nu_m+1)}{\Gamma(n+\mu_m+1)\Gamma(n+\nu_m+1)}}$$
$$\times(1-x)^{\frac{\mu_m}{2}}(1+x)^{\frac{\nu_m}{2}}P_n^{(\mu_m,\nu_m)}(x) \tag{3.13}$$

we impose the constraint (1.4) on equation (3.5) which gives

$$C_0=0,\qquad \gamma_{nm}^{\pm}=-\frac{1}{2}\pm\left(n+\frac{\mu_m+\nu_m+1}{2}\right)=\begin{cases}n+\dfrac{\mu_m+\nu_m}{2}=\gamma\\ -n-\dfrac{\mu_m+\nu_m}{2}-1=-\gamma-1\end{cases} \tag{3.14}$$

According to equation (3.2), which states that $\mu_m,\nu_m>0$, then $\gamma_{nm}^+>0$ and $\gamma_{nm}^-<-1$. Therefore, $\gamma>0$ and $E_\theta>0$ since $2E_\theta=\gamma(\gamma+1)$. Moreover, for a given value of γ (equivalently, E_θ), the above equation dictates that the integer m must belong to the set $0,\pm1,\pm2,\ldots,\pm j$ [or the set $\pm M,\pm(M+1),\pm(M+2),\ldots,\pm j$, see equation (3.3)] where the integer j satisfies the relation $\mu_j+\nu_j=2\gamma$. This statement is the analogue of that, for central potentials, which says that $m=0,\pm1,\pm2,\ldots,\pm\ell$, where ℓ is the orbital angular momentum quantum number. Additionally, for a given m the integer n should satisfy equation (3.14); that is, $n=\gamma-\frac{\mu_m+\nu_m}{2}$. This is also analogous to the central potential case where the principal quantum number is required to be equal to $\ell-|m|$ [7].

It is important to note that this diagonal representation is associated with the operator H_θ. That is, $(H_\theta)_{nn'}=E_\theta\delta_{nn'}$. One should not confuse this with the discrete bound states spectrum, which is associated with the diagonal representation of the total Hamiltonian H (i.e., $H_{nn'}=E\delta_{nn'}$). Consequently, it is neither required nor necessary to impose the constraints (3.14) on the bound states solution. That is, for bound states it is neither required that C_0 vanishes nor that γ (equivalently, E_θ) be quantized as $\gamma=(\mu_j+\nu_j)/2$. These points will be re-emphasized when we construct the complete solution space in sections 5 and 6.

3.2. Case (2.11b)

The second case (2.11*b*) is associated with the following angular component of the non-central potential:

$$V_\theta = \frac{\hat{C} + C\cos\theta}{2\sin^2\theta}, \tag{3.15}$$

where the real parameters μ and C_1 in (2.11*b*) are discretized via their relation to the integer m and the potential parameters as follows:

$$\mu_m = \sqrt{m^2 + \hat{C} + C}, \qquad C_1 = m^2 + \hat{C} - C, \tag{3.16}$$

which requires that, for real representations, $m^2 \geqslant -\hat{C} - C$. Thus, if we define M as the smallest integer greater than $\sqrt{|\hat{C} + C|}$, then

$$m = \begin{cases} 0, \pm 1, \pm 2, \ldots, & \hat{C} + C \geqslant 0 \\ \pm M, \pm(M+1), \pm(M+2), \ldots, & \hat{C} + C < 0. \end{cases} \tag{3.17}$$

It is also interesting to note that in this case, as well, a unique solution might exist for $m = \pm(M-1)$ in which the parameter μ_m is the negative of that given by equation (3.16) and its value will be in the range $(-1, 0)$. This happens for special values of the potential parameters $\hat{C}$ and C satisfying the following inequality:

$$1 - \hat{C} - C > (M-1)^2 > -\hat{C} - C. \tag{3.18}$$

However, the dimensionless parameter ν, aside from being larger than –1, is independent of m and still arbitrary. The tridiagonal matrix representation $\langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle$ for this case is

$$\begin{aligned} \langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle = {} & \left\{ \frac{C_1}{4} - \left(\frac{\nu+1}{2}\right)^2 - \frac{n(n+\mu)}{2n+\mu+\nu} + \frac{2n(n+\mu+\nu+1) + (\mu+\nu)(\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)} \right. \\ & \left. \times \left[\left(n + \frac{\mu+\nu}{2} + 1 \right)^2 - \tau^2 \right] \right\} \delta_{n,n'} + \frac{1}{2n+\mu+\nu} \\ & \times \sqrt{\frac{n(n+\mu)(n+\nu)(n+\mu+\nu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}} \left[\left(n + \frac{\mu+\nu}{2} \right)^2 - \tau^2 \right] \delta_{n,n'+1} \\ & + \frac{1}{2n+\mu+\nu+2} \sqrt{\frac{(n+1)(n+\mu+1)(n+\nu+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}} \\ & \times \left[\left(n + \frac{\mu+\nu}{2} + 1 \right)^2 - \tau^2 \right] \delta_{n,n'-1} \end{aligned} \tag{3.19}$$

where $\tau^2 = 2E_\theta + 1/4 = \left(\gamma + \frac{1}{2}\right)^2$ and, again, we took $2E_\theta = \gamma(\gamma+1)$. The subscript m on μ was removed here, as well, for the sake of clarity in presentation. Here again, to prevent the representation from degenerating and becoming indefinite, we require that the set of integers n and m satisfy the constraint $\left(n + \frac{\mu_m+\nu}{2} + 1\right)^2 \leqslant \tau^2$. That is,

$$n + \frac{\mu_m + \nu}{2} + 1 \leqslant \left| \gamma + \frac{1}{2} \right|. \tag{3.20}$$

Therefore, for a given γ and ν, $m = 0, \pm 1, \pm 2, \ldots, \pm j$ [or $m = \pm M, \pm(M+1), \pm(M+2), \ldots, \pm j$, see equation (3.17)], where j is the maximum integer that satisfies $\mu_j \leqslant |2\gamma + 1| - \nu - 2$. Moreover, $n = 0, 1, 2, \ldots, N$, where N is the maximum integer satisfying $N \leqslant \left|\gamma + \frac{1}{2}\right| - \frac{\mu_0+\nu}{2} - 1$ [or, $N \leqslant \left|\gamma + \frac{1}{2}\right| - \frac{\mu_M+\nu}{2} - 1$]. Expanding the angular

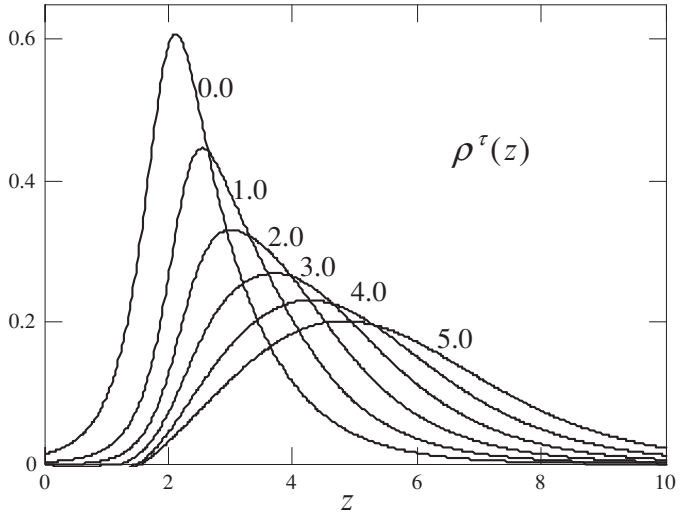

Figure 1. The density (weight) function $\rho^{\tau}(z)$ associated with the orthogonal polynomials $Q_n^{\tau}(z;\mu,\nu)$. The 'dispersion correction' method developed in [12] was used to generate this plot using the recursion coefficients $\{a_n, b_n\}_{n=0}^{50}$ obtained from equation (3.19) with $\mu = 1.0$, $\nu = 1.5$ and for several values of the parameter τ as shown on the traces.

component of the wavefunction as $\Theta(\theta) = \sum_n f_n(E_\theta)\chi_n(x;\mu,\nu)$ and defining the three-parameter orthogonal polynomials

$$Q_n^{\tau}(z;\mu,\nu) = \frac{1}{\sqrt{2n+\mu+\nu+1}}\sqrt{\frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}}\,f_n(z), \tag{3.21}$$

we obtain from equations (2.4*b*) and (3.19) the following three-term recursion relation that defines these polynomials:

$$\begin{aligned} zQ_n^{\tau} = {} & \left\{-\frac{n(n+\mu)}{2n+\mu+\nu} + \frac{2n(n+\mu+\nu+1)+(\mu+\nu)(\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\left[\left(n+\frac{\mu+\nu}{2}+1\right)^2-\tau^2\right]\right\}Q_n^{\tau} \\ & + \frac{(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)}\left[\left(n+\frac{\mu+\nu}{2}\right)^2-\tau^2\right]Q_{n-1}^{\tau} \\ & + \frac{(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)}\left[\left(n+\frac{\mu+\nu}{2}+1\right)^2-\tau^2\right]Q_{n+1}^{\tau} \end{aligned} \tag{3.22}$$

where $z = \left(\frac{\nu+1}{2}\right)^2 - \frac{C_1}{4}$. These polynomials are also new. However, comparing this recurrence relation with (A.1) of the Jacobi polynomials, we conclude that

$$P_n^{(\mu,\nu)}(z) = \lim_{\tau\to\infty} Q_n^{\tau}\left(-\tau^2\frac{1+z}{2};\mu,\nu\right). \tag{3.23}$$

The polynomials $Q_n^{\tau}(z;\mu,\nu)$ are uniquely defined by the recursion (3.22) and the initial normalizing relation that $Q_0^{\tau}(z;\mu,\nu) = 1$. Figure 1 gives a graphical illustration of the density function $\rho^{\tau}(z)$ associated with these polynomials for a given set of values of the parameters μ, ν and τ. The angular component of the wavefunction could then be written as

the L^2-series sum

$$\Theta_m^b(\theta) = \mathcal{N}_m^b(z) \sum_{n=0}^{N_m} \frac{2n+\mu+\nu+1}{\sqrt{2^{\mu+\nu+1}}} \frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)} \times Q_n^\tau(z;\mu,\nu)(1-x)^{\frac{\mu}{2}}(1+x)^{\frac{\nu+1}{2}} P_n^{(\mu,\nu)}(x), \tag{3.11b}$$

where $\mathcal{N}_m^b(z)$ is a normalization constant that depends on z, m and the physical parameters. Normalizability of $\Theta_m^b(\theta)$ in z space is achieved by taking $\mathcal{N}_m^b(z) = \sqrt{\rho^\tau(z)}$. For a given m, the integer index N_m of the last term in the series (3.11b) is equal to the largest integer n satisfying the constraint (3.20). Again, the presence of the subscript m on $\Theta_m^b(\theta)$ is due to the dependence of μ and z on m as given by equation (3.16).

The diagonal representation is similarly obtained by imposing the constraint (1.4) on equation (3.19), which gives

$$\nu = -1 + \sqrt{m^2 + \hat{C} - C} = -1 + \nu_m, \qquad \gamma_{nm}^{\pm} = -\frac{1}{2} \pm \left(n + \frac{\mu_m + \nu_m + 1}{2} \right), \tag{3.24}$$

where ν_m is given by equation (3.2). Consequently, this diagonal representation is equivalent to the case (2.11a) with $C_0 = 0$ and which is depicted by equations (3.14) and (3.13).

3.3. Case (2.11c)

Analysis of this solution space, which corresponds to the case (2.11c), shows that it is identical to the second case (2.11b) if we make the following replacements:

$$C_1 \to C_2, \qquad \mu \leftrightarrow \nu, \qquad \text{and} \quad x \to -x. \tag{3.25}$$

Consequently, we can directly write the L^2-series solution for this case as follows:

$$\Theta_m^c(\theta) = \mathcal{N}_m^c(z) \sum_{n=0}^{N_m} (-)^n \frac{2n+\mu+\nu+1}{\sqrt{2^{\mu+\nu+1}}} \frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)} \times Q_n^\tau(\tilde{z};\nu,\mu)(1-x)^{\frac{\mu+1}{2}}(1+x)^{\frac{\nu}{2}} P_n^{(\mu,\nu)}(x), \tag{3.11c}$$

where $\tilde{z} = \left(\frac{\mu+1}{2}\right)^2 - \frac{C_2}{4}$. The normalization constant $\mathcal{N}_m^c$ is obtained from $\mathcal{N}_m^b$ by the prescription (3.25) and we have used the Jacobi polynomials identity: $P_n^{(\mu,\nu)}(-x) = (-)^n \times P_n^{(\nu,\mu)}(x)$. The presence of the subscript m on $\Theta_m^c(\theta)$ is due to the dependence of ν and $\tilde{z}$ on m. The diagonal representation is obtained for the parameter values given by equation (3.24) with the exchange $\mu \leftrightarrow \nu$.

4. Solution space for the radial component

To obtain the total wavefunction we only need to calculate the remaining radial component $R(r)$ or, equivalently, its expansion coefficients $\{g_n\}_{n=0}^{\infty}$. To that end, we expand $|R\rangle$ in $(H-E)|R\rangle = 0$ as $\sum_m g_m|\xi_m\rangle$ and project on the left by $\langle\xi_n|$. This results in a three-term recursion relation that will be solved for g_n in terms of orthogonal polynomials. We only consider the situation described by (2.15a) which applies to the Coulomb potential and to all energies, positive and negative. If we rewrite the real dimensionless angular separation parameter E_θ as $2E_\theta = \gamma(\gamma+1)$, then the radial wave equation (2.4c) could be recast as follows:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\gamma(\gamma+1)}{r^2} - 2V_r + 2E \right] R = 0. \tag{4.1}$$

Hence, the parameter γ plays the role of the orbital angular momentum in problems with central (spherically symmetric) potentials. However, writing $2E_\theta$ as $\gamma(\gamma+1)$ implies that, for real values, E_θ is restricted to the range $E_\theta \geqslant -1/8$. Now, for positive values of E_θ, we either have $\gamma > 0$, which we refer to as γ^+, or $\gamma < -1$, which we call γ^-. For the case where the representation of the angular component is tridiagonal, $\gamma^\pm$ is a continuous parameter. However, for the diagonal angular representation it assumes discrete values indexed by n and m as given by equation (3.14). We start our investigation with the general continuous angular parameter γ. Consequently, the parameter α for the regular solution of the radial wavefunction could then be written in terms of $\gamma^\pm$ using (2.15a) as follows:

$$\alpha = \begin{cases} \gamma^+ + 1, & \gamma > 0 \\ -\gamma^-, & \gamma < -1 \end{cases} \tag{4.2}$$

which implies that α is always greater than +1. Now, to obtain the sought after recursion relation for the expansion coefficients of the radial wavefunction, we utilize the action of the wave operator on the basis as given by equation (2.14) and the parameter assignments (2.15a). This results (with the use of the orthogonality and recurrence relations of the Laguerre polynomials) in the following tridiagonal matrix representation

$$\begin{aligned} \frac{2}{\lambda^2}\langle \xi_k | H - E | \xi_{k'} \rangle = & \left[2(k+\alpha)\left(\frac{1}{4} - \frac{2E}{\lambda^2}\right) + \frac{2\mathcal{Z}}{\lambda} \right] \delta_{k,k'} \\ & + \left(\frac{1}{4} + \frac{2E}{\lambda^2}\right) [\sqrt{k(k+2\alpha-1)}\,\delta_{k,k'+1} + \sqrt{(k+1)(k+2\alpha)}\,\delta_{k,k'-1}], \end{aligned} \tag{4.3}$$

where $k, k' = 0, 1, 2, \ldots$ Therefore, the resulting recursion relation (1.3) for the expansion coefficients of the radial wavefunction becomes

$$\left[2(k+\alpha)\frac{\sigma_-}{\sigma_+} - \frac{2\mathcal{Z}/\lambda}{\sigma_+} \right] g_k - \sqrt{k(k+2\alpha-1)}\, g_{k-1} - \sqrt{(k+1)(k+2\alpha)}\, g_{k+1} = 0, \tag{4.4}$$

where $\sigma_\pm = \frac{2E}{\lambda^2} \pm \frac{1}{4}$. Rewriting this recursion in terms of the polynomials $P_k(E) = \sqrt{\Gamma(k+2\alpha)/\Gamma(k+1)}\, g_k(E)$, we obtain a more familiar recursion relation as follows:

$$2\left[(k+\alpha)\frac{\sigma_-}{\sigma_+} - \frac{\mathcal{Z}/\lambda}{\sigma_+} \right] P_k - (k+2\alpha-1) P_{k-1} - (k+1) P_{k+1} = 0. \tag{4.5}$$

We compare this with the three-term recursion relation for the Meixner–Pollaczek polynomials $P_k^\mu(z,\varphi)$ [8]:

$$2[(k+\mu)\cos\varphi + z\sin\varphi] P_k^\mu - (k+2\mu-1) P_{k-1}^\mu - (k+1) P_{k+1}^\mu = 0, \tag{4.6}$$

where $\mu > 0$ and $0 < \varphi < \pi$. The comparison is valid only within this permissible range of values of the parameters. This means that the solution obtained as such is valid only for $E > 0$ (i.e., for the continuum scattering states). Thus for the continuum case, we obtain the following two alternative L^2-series solutions (depending on the value of the parameter γ) for the radial component of the wavefunction

$$\begin{aligned} R^{\gamma^+}(r) = & \mathcal{N}^{\gamma^+}(E) \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+2\gamma^++2)} P_k^{\gamma^++1}\left(\frac{-\mathcal{Z}}{\sqrt{2E}}, \cos^{-1}\frac{E-\lambda^2/8}{E+\lambda^2/8} \right) \\ & \times (\lambda r)^{\gamma^++1} \mathrm{e}^{-\lambda r/2} L_k^{2\gamma^++1}(\lambda r), \end{aligned} \tag{4.7a}$$

$$\begin{aligned} R^{\gamma^-}(r) = & \mathcal{N}^{\gamma^-}(E) \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-2\gamma^-)} P_k^{-\gamma^-}\left(\frac{-\mathcal{Z}}{\sqrt{2E}}, \cos^{-1}\frac{E-\lambda^2/8}{E+\lambda^2/8} \right) \\ & \times (\lambda r)^{-\gamma^-} \mathrm{e}^{-\lambda r/2} L_k^{-2\gamma^--1}(\lambda r), \end{aligned} \tag{4.7b}$$

where $\mathcal{N}^{\gamma^{\pm}}$ are normalization constants that depend only on the physical parameters of the problem. To make $R^{\gamma^{\pm}}(r)$ energy normalizable we write $\mathcal{N}^{\gamma} = \sqrt{\lambda \mathcal{Z} \rho^{\gamma}(z)/(2E)^{3/2}}$, where $\rho^{\gamma}(z)$ is the density function associated with the orthogonality of the Meixner–Pollaczek polynomials:

$$\int \rho^{\mu}(z) P_n^{\mu}(z,\varphi) P_m^{\mu}(z,\varphi)\,\mathrm{d}z = \frac{\Gamma(n+2\mu)}{\Gamma(n+1)}\delta_{nm}. \tag{4.8}$$

On the other hand, imposing the diagonalization constraints (1.4) on the tridiagonal matrix representation (4.3) gives the following energy spectrum:

$$E_k = -\mathcal{Z}^2/(k+\alpha)^2 = \begin{cases} -\mathcal{Z}^2/(k+\gamma^{+}+1)^2 \\ -\mathcal{Z}^2/(k-\gamma^{-})^2 \end{cases} \tag{4.9}$$

and requires that the length scale parameter λ be discretized as follows:

$$\lambda_k = -2\mathcal{Z}/(k+\alpha) = \begin{cases} -2\mathcal{Z}/(k+\gamma^{+}+1) \\ -2\mathcal{Z}/(k-\gamma^{-}) \end{cases} \tag{4.10}$$

which requires that $\mathcal{Z} < 0$ since λ must be positive (i.e., bound states exist only for an attractive coulomb potential). The corresponding radial component of the discrete bound states wavefunction is

$$R_k^{+}(r) = \xi_k\big(y;\lambda_k^{+},\nu^{+}\big) = \sqrt{\frac{\lambda_k^{+}\Gamma(k+1)}{\Gamma(k+2\gamma^{+}+2)}}\big(\lambda_k^{+}r\big)^{\gamma^{+}+1}\,\mathrm{e}^{-\lambda_k^{+}r/2}L_k^{2\gamma^{+}+1}\big(\lambda_k^{+}r\big), \tag{4.11a}$$

$$R_k^{-}(r) = \xi_k\big(y;\lambda_k^{-},\nu^{-}\big) = \sqrt{\frac{\lambda_k^{-}\Gamma(k+1)}{\Gamma(k-2\gamma^{-})}}\big(\lambda_k^{-}r\big)^{-\gamma^{-}}\,\mathrm{e}^{-\lambda_k^{-}r/2}L_k^{-2\gamma^{-}-1}\big(\lambda_k^{-}r\big), \tag{4.11b}$$

where $\lambda_k^{\pm}$ is given by equation (4.10) above.

Now, if the angular representation is diagonal, then $\gamma^{\pm}$ assumes the discrete values given by equation (3.14). In this case, the radial component of the wavefunction for the continuum scattering states could be written as

$$R_{nm}(r) = \mathcal{N}_{nm}(E)\sum_{k=0}^{\infty}\left[\frac{\Gamma(k+1)}{\Gamma(k+2n+\mu_m+\nu_m+2)}P_k^{n+\frac{\mu_m+\nu_m}{2}+1}\left(\frac{-\mathcal{Z}}{\sqrt{2E}},\cos^{-1}\frac{E-\lambda^2/8}{E+\lambda^2/8}\right)\right.$$
$$\left.\times\,(\lambda r)^{n+\frac{\mu_m+\nu_m}{2}+1}\,\mathrm{e}^{-\lambda r/2}L_k^{2n+\mu_m+\nu_m+1}(\lambda r)\right]. \tag{4.12}$$

Whereas, for bound states, it reads

$$R_{knm}(r) = \xi_k(y;\lambda_{knm},\nu_{knm})$$
$$= \sqrt{\frac{\lambda_{knm}\Gamma(k+1)}{\Gamma(k+2n+\mu_m+\nu_m+2)}}(\lambda_{knm}r)^{n+\frac{\mu_m+\nu_m}{2}+1}\,\mathrm{e}^{-\lambda_{knm}r/2}L_k^{2n+\mu_m+\nu_m+1}(\lambda_{knm}r) \tag{4.13}$$

where $\lambda_{knm} = -2\mathcal{Z}\big/\big(k+n+\frac{\mu_m+\nu_m}{2}+1\big)$ and the corresponding energy spectrum is

$$E_{knm} = -\mathcal{Z}^2\bigg/\left(k+n+\frac{\mu_m+\nu_m}{2}+1\right)^2. \tag{4.14}$$

In the following two sections we use the above findings to construct the complete solution space for the continuum scattering states and for the discrete bound states. Two alternative solutions are obtained depending on whether or not the potential parameter C_0 vanishes.

5. The complete solution space for $C_0 \neq 0$

For the continuum scattering states, this solution space is parametrized by three quantum numbers and six real parameters. The real parameters are $\{\hat{C}, C, C_0, \mathcal{Z}, \gamma, E\}$, where $E > 0$ and either $\gamma = \gamma^+ > 0$ or $\gamma = \gamma^- < -1$. The quantum numbers are $\{k, n, m\}$, where $k \in \mathbb{N} = 0, 1, 2, \ldots$, and $m \in \mathbb{Z}^j = 0, \pm 1, \ldots, \pm j$ [or $m = \pm M, \pm(M+1), \pm(M+2), \ldots, \pm j$, see equation (3.3)]. j is the largest integer satisfying $\mu_j + \nu_j \leqslant 2\gamma^+$ or $\mu_j + \nu_j \leqslant -2(\gamma^- + 1)$ and M is the smallest integer greater than $\max(|\hat{C} \pm C|)^{1/2}$. For a given integer m, $n = 0, 1, 2, \ldots, N_m$, where N_m is the largest integer n that satisfies either $n \leqslant \gamma^+ - \frac{\mu_m + \nu_m}{2}$ or $n \leqslant -\gamma^- - 1 - \frac{\mu_m + \nu_m}{2}$. It might be worthwhile to compare this with the spherically symmetric Coulomb problem [1, 9] whose continuum solution space is parametrized by four quantum numbers $\{k, n, m, \ell\}$ and two real parameters $\{\mathcal{Z}, E\}$, where $E > 0$, $k, \ell \in \mathbb{N}$ and $m \in \mathbb{Z}^\ell = 0, \pm 1, \ldots, \pm \ell$ with $n = 0, 1, 2, \ldots, \ell - |m|$. Now, the total wavefunction, which is an element in this space, is written in terms of the radial component $R^{\gamma^\pm}(r)$ in (4.7a), (4.7b) and the angular components, $\Theta_m^a(\theta)$ in (3.11a) and $\Phi_m(\phi)$ in (2.6).

For bound states, the energy spectrum is quantized via its dependence on a single quantum number as given by equation (4.9) and the complete bound states wavefunction is written in terms of the angular components, $\Phi_m(\phi)$ in (2.6) and $\Theta_m^a(\theta)$ in (3.11a), and the radial component in (4.11a), (4.11b).

It is worthwhile to consider the unique case where $\hat{C} = C = 0$ and $C_0 \neq 0$ which is a special finding of our work. In this case, equation (3.2) gives $\mu_m = \nu_m = |m|$. If we write the positive (negative) parameter γ as $\gamma^+ = j + \eta$ ($\gamma^- = -j - 1 - \eta$), where $0 \leqslant \eta < 1$, then the complete scattering state wavefunction becomes

$$\psi_j(\vec{r}, E) = \mathcal{N} \sum_{k=0}^{\infty} \sum_{n=0}^{j} \sum_{m=n-j}^{j-n} \left[\frac{(n + |m| + 1/2)\Gamma(k+1)}{2^{|m|}\Gamma(k + 2j + 2\eta + 2)} \frac{\Gamma(n+1)\Gamma(n + 2|m| + 1)}{\Gamma(n + |m| + 1)^2} \right.$$
$$\times P_k^{j+\eta+1}\left(\frac{-\mathcal{Z}}{\sqrt{2E}}, \cos^{-1}\frac{E - \lambda^2/8}{E + \lambda^2/8}\right) \times H_n^{|m|}(z; C_0)$$
$$\left. \times (\lambda r)^{j+\eta+1} \mathrm{e}^{-\lambda r/2}(1 - x^2)^{\frac{|m|}{2}} \times L_k^{2j+2\eta+1}(\lambda r) \times P_n^{(|m|,|m|)}(x) \times \mathrm{e}^{\mathrm{i}m\phi} \right] \tag{5.1}$$

where the normalization constant $\mathcal{N} = \sqrt{\lambda \mathcal{Z} \rho^\sigma(z) \rho^\gamma\left(\frac{-\mathcal{Z}}{\sqrt{2E}}\right) \big/ \pi (2E)^{3/2}}$, $z = \frac{1}{C_0}\left(\eta + j + \frac{1}{2}\right)^2$ and the polynomials $H_n^{|m|}(z; C_0)$ satisfy the three-term recursion relation

$$z H_n^{|m|} = \frac{1}{C_0}\left(n + |m| + \frac{1}{2}\right)^2 H_n^{|m|} + \frac{n + |m|}{2(n + |m| + 1/2)} H_{n-1}^{|m|}$$
$$+ \frac{(n+1)(n + 2|m| + 1)}{2(n + |m| + 1/2)(n + |m| + 1)} H_{n+1}^{|m|}. \tag{3.9$'$}$$

The bound states energy spectrum for this special case is $E_{kj} = -\mathcal{Z}^2/(k + j + \eta + 1)^2$ and the corresponding wavefunction is

$$\psi_{kj}(\vec{r}) = \mathcal{N}_k \sum_{n=0}^{j} \sum_{m=n-j}^{j-n} \left[\frac{(n + |m| + 1/2)}{2^{|m|}} \frac{\Gamma(n+1)\Gamma(n + 2|m| + 1)}{\Gamma(n + |m| + 1)^2} H_n^{|m|}(z; C_0) \right.$$
$$\left. \times (\lambda_{kj} r)^{j+\eta+1} \mathrm{e}^{-\lambda_{kj} r/2}(1 - x^2)^{\frac{|m|}{2}} \times L_k^{2j+2\eta+1}(\lambda_{kj} r) \times P_n^{(|m|,|m|)}(x) \times \mathrm{e}^{\mathrm{i}m\phi} \right] \tag{5.2}$$

where $\lambda_{kj} = -2\mathcal{Z}/(k + j + \eta + 1)$ and $\mathcal{N}_k = \sqrt{\lambda \rho^\sigma(z) \Gamma(k+1) / \pi \Gamma(k + 2j + 2\eta + 2)}$.

6. The complete solution space for $C_0 = 0$

This space splits into four subspaces depending on the values of the potential parameters, $\hat{C}$ and C:

6.1. $\hat{C} \pm C \geqslant 0$

The angular component of the solution space carries the diagonal representation $\Theta^a_{nm}(\theta) = \chi_n(x; \mu_m, \nu_m)$ given by equation (3.13), which is associated with the case (2.11a), where $m = 0, \pm 1, \pm 2, \ldots, \pm j$, $\mu_j + \nu_j = 2\gamma$ and, for a given m, $n = \gamma - \frac{\mu_m + \nu_m}{2}$. The other angular component of the wavefunction is $\Phi_m(\phi)$ which is given by (2.6). For the continuum scattering states, the radial component is given by equation (4.12), whereas, for the discrete bound states it is given by equation (4.13) and the corresponding energy spectrum is given by equation (4.14).

6.2. $\hat{C} \pm C < 0$

This case is identical to the previous one except that the angular phase quantum number m belongs to the range of values $m = \pm M, \pm(M+1), \pm(M+2), \ldots, \pm j$, where M is the smallest integer that is greater than $\max(|\hat{C} \pm C|)^{1/2}$.

Almost all of the work that has been reported in the literature about exact solutions of this kind of non-central problems (with $V_\phi = 0$) is confined to either one of the above two representations, where the angular component is diagonal, $\alpha = \mu_m/2$, $\beta = \nu_m/2$, and $C_0 = 0$. Moreover, some of these solutions were obtained using path integral formulation utilizing the Kustannheimo–Stiefel transformation. Additionally, most are dealing with bound states while others were solved only in parabolic coordinates. For more recent examples of such work, one may consult the papers in [10] and references therein. The article by Khare and Bhaduri [11] is a good introduction to the general problem of non-central potentials in two and three dimensions.

6.3. $C > \hat{C} \geqslant -C$

This solution space carries the tridiagonal representation $\Theta^b_m(\theta)$ given by equation (3.11b), which is associated with the case (2.11b), where $m = 0, \pm 1, \pm 2, \ldots, \pm j$ and j is the maximum integer that satisfies $\mu_j \leqslant |2\gamma + 1| - \nu - 2$. Moreover, for a given γ and ν, N_m is equal to the largest integer n satisfying $n \leqslant \left|\gamma + \frac{1}{2}\right| - \frac{\mu_m + \nu}{2} - 1$. The other angular component $\Phi_m(\phi)$ is given by (2.6). The radial component for the continuum scattering states is given by equations (4.7a), (4.7b), whereas, for the discrete bound states it is given by equations (4.11a), (4.11b) and the corresponding energy spectrum is that given by equation (4.9).

6.4. $-C > \hat{C} \geqslant C$

The angular component of this solution space is tridiagonal and corresponds to the case (2.11c). Consequently, it is identical to the previous case after the application of the map given by (3.25). For example, j is the maximum integer that satisfies $\nu_j \leqslant |2\gamma + 1| - \mu - 2$ and N_m is the largest integer n satisfying $n \leqslant \left|\gamma + \frac{1}{2}\right| - \frac{\nu_m + \mu}{2} - 1$.

7. Solution for the radial oscillator potential

In this section we give a brief treatment of the same problem but with the radial component of the non-central potential (2.3) being the oscillator potential $V_r = \frac{1}{2}\omega^4 r^2$, where ω is the oscillator frequency. This is accomplished by taking the configuration space coordinate $y = (\lambda r)^2$ in the expression of the L^2 basis elements (2.12). Using the differential equation and differential formula for the Laguerre polynomials, the action of the wave operator of equation (2.4c) with $2E_\theta = \gamma(\gamma+1)$ on this basis gives the following:

$$(H-E)\xi_n = 2\lambda^2\Bigg[-\frac{n\left(2\alpha-\nu-\frac{1}{2}\right)+\left(\alpha-\frac{1}{4}\right)^2-\frac{1}{4}\left(\gamma+\frac{1}{2}\right)^2}{y}+n+\alpha+\frac{1}{4}-\frac{y}{4} + \frac{1}{2\lambda^2}(V_r-E)\Bigg]\xi_n + 2\lambda^2\frac{(n+\nu)\left(2\alpha-\nu-\frac{1}{2}\right)}{y}\frac{A_n}{A_{n-1}}\xi_{n-1}. \tag{7.1}$$

The recurrence relation and orthogonality property of the Laguerre polynomials show that a tridiagonal matrix representation $\langle\xi_n|\, H-E\,|\xi_{n'}\rangle$ is possible only for a limited number of special radial potential components V_r and results in the following two possibilities:

$$(1)\quad \nu = 2\alpha - 1/2 \qquad 2\alpha = \begin{cases}\gamma^+ + 1\\ -\gamma^-\end{cases} \qquad \text{and} \quad V_r = \tfrac{1}{2}\omega^4 r^2 \tag{7.2a}$$

$$(2)\quad \nu = 2\alpha - 3/2 \qquad \text{and} \qquad V_r = \tfrac{1}{2}\lambda^4 r^2 + \mathcal{B}/2r^2 \tag{7.2b}$$

where ω is the oscillator frequency and $\mathcal{B}$ is a centripetal potential barrier parameter. We only consider the first case (7.2a) which results in the following tridiagonal matrix representation of the wave operator:

$$\frac{2}{\lambda^2}\langle\xi_k|H-E|\xi_{k'}\rangle = \left[(2k+\nu+1)\left(\frac{\omega^4}{\lambda^4}+1\right)-\frac{2E}{\lambda^2}\right]\delta_{k,k'} - \left(\frac{\omega^4}{\lambda^4}-1\right)\left[\sqrt{k(k+\nu)}\,\delta_{k,k'+1}+\sqrt{(k+1)(k+\nu+1)}\,\delta_{k,k'-1}\right]. \tag{7.3}$$

This representation leads to a three-term recursion relation for the expansion coefficients of the radial component of the wavefunction which could be solved in terms of a 'hyperbolic-type' Meixner–Pollaczek polynomial defined as

$$\tilde{P}_k^\mu(z,\varphi) \equiv P_k^\mu(-\mathrm{i}z,\mathrm{i}\varphi) = \frac{\Gamma(k+2\mu)}{\Gamma(k+1)\Gamma(2\mu)}\,\mathrm{e}^{-k\varphi}\,{}_2F_1(-k,\mu+z;2\mu;1-\mathrm{e}^{2\varphi}), \tag{7.4}$$

where $\varphi > 0$ and ${}_2F_1$ is the hypergeometric function. These polynomials satisfy the following modified (with respect to the original relation (4.6)) three-term recursion relation:

$$2[(k+\mu)\cosh\varphi + z\sinh\varphi]\tilde{P}_k^\mu - (k+2\mu-1)\tilde{P}_{k-1}^\mu - (k+1)\tilde{P}_{k+1}^\mu = 0. \tag{7.5}$$

However, we will not pursue this line of investigation and be content with only obtaining the discrete bound state wavefunction and energy spectrum. This requires that the matrix representation of the Hamiltonian be diagonal, which when imposed on (7.3) gives $\lambda^2 = \omega^2$ and the following energy spectrum:

$$E_k^\pm = \begin{cases}\omega^2(2k+\gamma^+ + 3/2)\\ \omega^2(2k-\gamma^- + 1/2).\end{cases} \tag{7.6}$$

The total bound states wavefunction for the case $C_0 \neq 0$ is written in terms of the angular components $\Phi_m(\phi)$ in (2.6), $\Theta_m^a(\theta)$ in (3.11a) and the following radial component:

$$R_k^+(r) = \sqrt{\frac{2\omega\Gamma(k+1)}{\Gamma(k+\gamma^+ + 3/2)}}(\omega r)^{\gamma^+ + 1}\,\mathrm{e}^{-\omega^2 r^2/2} L_k^{\gamma^+ + 1/2}(\omega^2 r^2), \tag{7.7a}$$

$$R_k^-(r) = \sqrt{\frac{2\omega\Gamma(k+1)}{\Gamma(k-\gamma^- + 1/2)}}(\omega r)^{-\gamma^-}\,\mathrm{e}^{-\omega^2 r^2/2} L_k^{-\gamma^- - 1/2}(\omega^2 r^2). \tag{7.7b}$$

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Appendix A. The Laguerre and Jacobi polynomials

The following are useful formulae and relations associated with these polynomials. They are found in most books on orthogonal polynomials but listed here for ease of reference.

(1) The Jacobi polynomials $P_n^{(\mu,\nu)}(x)$, where $\mu > -1$, $\nu > -1$:

$$\begin{aligned}\left(\frac{1 \pm x}{2}\right) P_n^{(\mu,\nu)} &= \frac{2n(n+\mu+\nu+1) + (\mu+\nu)\left(\frac{\mu+\nu}{2} \pm \frac{\nu-\mu}{2} + 1\right)}{(2n+\mu+\nu)(2n+\mu+\nu+2)} P_n^{(\mu,\nu)} \\ &\quad \pm \frac{(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)} P_{n-1}^{(\mu,\nu)} \\ &\quad \pm \frac{(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)} P_{n+1}^{(\mu,\nu)}\end{aligned} \tag{A.1}$$

$$\begin{aligned}P_n^{(\mu,\nu)}(x) &= \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)\Gamma(\mu+1)}\,{}_2F_1\left(-n, n+\mu+\nu+1; \mu+1; \frac{1-x}{2}\right) \\ &= (-)^n P_n^{(\nu,\mu)}(-x)\end{aligned} \tag{A.2}$$

$$\left\{(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - [(\mu+\nu+2)x + \mu - \nu]\frac{\mathrm{d}}{\mathrm{d}x} + n(n+\mu+\nu+1)\right\} P_n^{(\mu,\nu)}(x) = 0 \tag{A.3}$$

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\mu,\nu)} = -n\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right) P_n^{(\mu,\nu)} + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu} P_{n-1}^{(\mu,\nu)} \tag{A.4}$$

$$\begin{aligned}&\int_{-1}^{+1} (1-x)^\mu (1+x)^\nu P_n^{(\mu,\nu)}(x) P_m^{(\mu,\nu)}(x)\,\mathrm{d}x \\ &\qquad = \frac{2^{\mu+\nu+1}}{2n+\mu+\nu+1}\frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}\delta_{nm}.\end{aligned} \tag{A.5}$$

(2) The Laguerre polynomials $L_n^\nu(x)$, where $\nu > -1$:

$$xL_n^\nu = (2n+\nu+1)L_n^\nu - (n+\nu)L_{n-1}^\nu - (n+1)L_{n+1}^\nu \tag{A.6}$$

$$L_n^\nu(x) = \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(\nu+1)}\,{}_1F_1(-n; \nu+1; x) \tag{A.7}$$

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\nu+1-x)\frac{\mathrm{d}}{\mathrm{d}x} + n\right] L_n^\nu(x) = 0 \tag{A.8}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}L_n^\nu = nL_n^\nu - (n+\nu)L_{n-1}^\nu \tag{A.9}$$

$$\int_0^\infty x^\nu \,\mathrm{e}^{-x} L_n^\nu(x) L_m^\nu(x)\,\mathrm{d}x = \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)}\delta_{nm}. \tag{A.10}$$

Appendix B. The orthogonal polynomials $H_n^\sigma(z;\mu,\nu)$ and $Q_n^\tau(z;\mu,\nu)$

For the purpose of economy in presentation we make a simultaneous treatment of both polynomials by studying the expansion coefficients (polynomials) $\{f_n(z)\}_{n=0}^\infty$ which are related to these two polynomials by the definitions (3.8) and (3.21), respectively. The recursion relation (1.3) together with the initial seed

$$f_0 = \sqrt{\mu+\nu+1}\sqrt{\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}}, \tag{B.1}$$

which is obtained from the normalization $H_0^\sigma = Q_0^\tau = 1$, gives a unique definition of these polynomials. In fact, one obtains recursively

$$\begin{aligned} f_1(z) &= \frac{z-a_0}{b_0} f_0 \\ f_2(z) &= \frac{1}{b_1}[(z-a_1)f_1(z) - b_0 f_0] \\ f_3(z) &= \frac{1}{b_2}[(z-a_2)f_2(z) - b_1 f_1(z)] \\ &\cdots. \end{aligned} \tag{B.2}$$

These polynomials are orthonormal with respect to the measure $\rho(z)\,\mathrm{d}z$, where $\rho(z)$ is the weight (density) function associated with these polynomials. That is

$$\int_{z_-}^{z_+} \rho(z) f_n(z) f_m(z)\,\mathrm{d}z = \delta_{nm}, \tag{B.3a}$$

where $z_\mp$ is the left/right boundary of the one-dimensional real space with coordinate z. This orthogonality relation could also be written in terms of the target polynomials (e.g., H_n^σ) as follows:

$$\int_{z_-}^{z_+} \rho^\sigma(z) H_n^\sigma(z;\mu,\nu) H_m^\sigma(z;\mu,\nu)\,\mathrm{d}z = \frac{1}{2n+\mu+\nu}\frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}\delta_{nm}. \tag{B.3b}$$

One way to obtain the density function is to use its relationship to the resolvent operator (Green's function), $G(z)$, associated with this system of polynomials:

$$\rho(z) = \frac{1}{2\pi\,\mathrm{i}}[G(z+\mathrm{i}0^+) - G(z-\mathrm{i}0^-)] = \frac{1}{\pi}\,\mathrm{Im}\,G(z+\mathrm{i}0^+). \tag{B.4}$$

That is, the density is equal to the discontinuity of the Green's function across the cut on the real z-axis in the complex plane. One of the representations of the Green's function could be written in terms of the recursion coefficients $\{a_n, b_n\}_{n=0}^\infty$ as the infinite continued fraction:

$$G(z) = \cfrac{-1}{z - a_0 - \cfrac{b_0^2}{z-a_1 - \cfrac{b_1^2}{z-a_2 \cdots}}}. \tag{B.5}$$

Assuming that the system under study has a single density band with finite width, then there exist a single limit for the recursion coefficients:

$$a_\infty = \lim_{n\to\infty} a_n, \qquad b_\infty = \lim_{n\to\infty} b_n. \tag{B.6}$$

In such a case the boundary of the band (the ends of the non-zero support interval in the real space z) is

$$z_{\pm} = a_{\infty} \pm 2b_{\infty}, \tag{B.7}$$

and the resolvent operator (B.5) could be approximated by

$$G(z) \approx \frac{-1}{z - a_0 - \dfrac{b_0^2}{z-a_1-\dfrac{b_1^2}{z-a_2\cdots-\dfrac{b_{N-2}^2}{z-a_{N-1}-T(z)}}}}. \tag{B.8}$$

for some large enough integer N such that $|a_{\infty} - a_N| \approx 0$, $|b_{\infty} - b_{N-1}| \approx 0$. The 'terminator' function $T(z)$ is given by

$$T(z) = \frac{b_{\infty}^2}{z - a_{\infty} - \frac{b_{\infty}^2}{z-a_{\infty}-\cdots}} = \frac{b_{\infty}^2}{z - a_{\infty} - T(z)} \tag{B.9}$$

which could be solved for $T(z)$ as

$$T(z) = \frac{z - a_{\infty}}{2} \pm \frac{1}{2}\sqrt{(z - a_{\infty} - 2b_{\infty})(z - a_{\infty} + 2b_{\infty})}. \tag{B.10}$$

Therefore, it becomes obvious from this expression that $\operatorname{Im} T(z) = 0$ results in $\rho(z) = 0$ outside the density band and gives its boundaries as $a_{\infty} \pm 2b_{\infty}$. However, in the present case such limits do not exist. Specifically, for a large integer N we obtain from equation (3.7) the following values associated with the polynomials $\{H_n^{\sigma}\}$:

$$a_N \approx \sigma N^2, \qquad b_N \approx 1/2, \tag{B.11}$$

whereas, for the polynomials $\{Q_n^{\tau}\}$, these could be obtained from (3.19) as follows:

$$a_N \approx N^2/2, \qquad b_N \approx N^2/4. \tag{B.12}$$

Consequently, we could infer that the interval on the real z-axis associated with the polynomials $\{Q_n^{\tau}\}$ is the semi-infinite $z \in [0, \infty]$, whereas for $\{H_n^{\sigma}\}$ the corresponding interval is σ-dependent (for $\sigma \to 0$, $z \in [-1, +1]$). These results are also supported by the relation of these polynomials to the Jacobi polynomial (equations (3.23) and (3.10), respectively) which is defined in the interval $-1 \leqslant x \leqslant 1$. To give a fairly accurate graphical representation of the weight function we use one of three numerical methods developed in [12] for obtaining a good approximation of the density function associated with finite tridiagonal Hamiltonian matrices. As an example, we consider $\rho^{\tau}(z)$ associated with the orthogonal polynomials $\{Q_n^{\tau}\}$. Figure 1 shows this density function for a given value of μ and ν and for several choices of the parameter τ.

Appendix C. The Aharonov–Bohm and magnetic monopole

In this appendix, we consider briefly the combined Aharonov–Bohm (A–B) effect and a magnetic monopole for which the electromagnetic vector potential $\vec{A}$ has the following components in spherical coordinates:

$$A_r = 0, \qquad A_{\theta} = 0, \qquad A_{\phi} = \frac{a - b\cos\theta}{r\sin\theta}, \tag{C.1}$$

where a and b are real parameters and the A–B magnetic flux strength is $2\pi|a - b|$. The monopole strength is b and it has a singularity along the negative z-axis. The wave equation

of a charged particle in this electromagnetic potential is obtained by replacing equation (2.1) by another with minimal coupling to the vector potential as follows:

$$\left[-\tfrac{1}{2}(\vec{\nabla} - \mathrm{i}\vec{A})^2 + V(\vec{r}) - E\right]\psi = 0, \tag{C.2}$$

where we took $e/\hbar c = 1$. If we also consider the contribution of a static charge $\mathcal{Z}e$ fixed at the origin, then $V(\vec{r}) = \mathcal{Z}/r$ and the total potential field experienced by the charged particle, whose dynamics is described by equation (C.2), is non-central and could be written as

$$V(r,\theta) = \frac{\mathcal{Z}}{r} - \frac{b^2/2}{r^2} + \frac{1}{r^2\sin^2\theta}\left[\frac{a^2+b^2}{2} - ma + b(m-a)\cos\theta\right], \tag{C.3}$$

where we have used the ϕ-dependence of the complete wavefunction ψ as $\mathrm{e}^{\mathrm{i}m\phi}$ when writing $\partial_\phi\psi = \mathrm{i}m\psi$. Comparing this potential with (2.3), we conclude that

$$V_r = \frac{\mathcal{Z}}{r} - \frac{b^2/2}{r^2}, \qquad C_0 = 0, \qquad \hat{C} = a^2 + b^2 - 2ma, \qquad C = 2b(m-a). \tag{C.4}$$

Consequently, this situation is compatible with the radial potential configuration given by the case (2.15*b*), where the centrifugal barrier parameter $\mathcal{B} = -b^2$ and negative. Moreover, equation (3.2) gives $\mu_m = |a-b-m|$ and $\nu_m = |a+b-m|$. Using the findings in section 2 of [1] where the case (2.15*b*) is discussed, we can directly obtain the bound states energy spectrum as

$$E_{knm} = -\mathcal{Z}^2/2(k+\nu_{nm}+1)^2, \tag{C.5}$$

where $\nu_{nm} = -\frac{1}{2} + \sqrt{\left(\gamma^{\pm}_{nm} + \frac{1}{2}\right)^2 - b^2}$ and $\gamma^{\pm}_{nm}$ is given by equation (3.14). That is, we obtain

$$E_{knm} = -\mathcal{Z}^2 \Bigg/ 2\left[k + \frac{1}{2} + \sqrt{\left(n + \frac{1}{2}|a-b-m| + \frac{1}{2}|a+b-m| + \frac{1}{2}\right)^2 - b^2}\right]^2. \tag{C.6}$$

This agrees with the results found in [13]. It should be noted that the expression under the square root is always positive for all real values of a and b and for all integers n and m. The radial component of the complete bound states wavefunction could also be obtained using results from the same work cited above [1] as

$$R_{knm}(r) = \sqrt{\frac{\lambda_{knm}\Gamma(k+1)}{\Gamma(k+2\nu_{nm}+1)}}(\lambda_{knm}r)^{1+\nu_{nm}}\,\mathrm{e}^{-\lambda_{knm}r/2}L_k^{2\nu_{nm}}(\lambda_{knm}r), \tag{C.7}$$

where $\lambda_{knm} = 2\sqrt{-2E_{knm}} = 2|\mathcal{Z}|/(k+\nu_{nm}+1)$. The angular components of the complete wavefunction belong to one of the cases given in section 6. The choice depends on the values of the potential parameters, a and b. The pure A–B effect takes place in the absence of the monopole (i.e., for $b = 0$), which also imply zero centrifugal barrier.

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